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Journal of Computational and Applied Mathematics 199 (2007) 445–451

JOURNAL OF COMPUTATIONAL AND APPLIED MATHEMATICS

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Numerical verification of stationary solutions for Navier–Stokes problems

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Received 1 February 2005

Abstract

We present a numerical method to enclose stationary solutions of the Navier–Stokes equations, especially 2-D driven cavity problem with regularized boundary condition. Our method is based on the infinite dimensional Newton's method by estimating the inverse of the corresponding linearized operator. The method can be applied to the case for high Reynolds numbers and we show some numerical examples which confirm us the actual effectiveness.

Keywords: Numerical enclosure method; Driven cavity flows; Infinite dimensional Newton's method

1. Introduction

We consider the following Navier–Stokes equations

$$\begin{cases} -\Delta u + R\cdot(u\cdot\nabla)u + \nabla p = f & \text{in } \Omega, \\ \operatorname{div} u = 0 & \text{in } \Omega, \\ u = g & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where u, p and R are the velocity vector, pressure and the Reynolds number, respectively and the flow region Ω is a convex polygonal domain in $\mathbf{R}^2$. In what follows, for each rational number m, let $H^m(\Omega)$ denote the L^2-Sobolev space of order m on Ω. The function $f=(f_1, f_2)$ means a density of body forces with $f \in (H^1(\Omega))^2$ and $g=(g_1, g_2) \in (H^{1/2}(\partial\Omega))^2$, where we assume that there exists a function $\varphi \in H^2(\Omega)$ satisfying $(\varphi_y, -\varphi_x) = g$ on $\partial\Omega$.

The above problem was discussed by Wieners [7] for low Reynolds numbers. The method proposed in it is based on Newton–Kantorovich theorem but it would not be able to apply to high Reynolds numbers, because the estimation for the inverse of the linearized operator directly depends on the Reynolds number. We also uses Newton type verification condition, but the method which verifies the invertibility of linearized operator is different from the Wieners' formulation. Our method has an advantage which enables us to verify the invertibility of the linearized operator,

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0377-0427/$ - see front matter
doi:10.1016/j.cam.2005.09.031

even for high Reynolds numbers, provided that the approximation subspace is sufficiently accurate and that the inverse operator actually exists in the rigorous sense. The numerical examples presented in Section 5 show this actual improvement.

2. Stream function and the linearized operator

We first introduce a stream function ψ satisfying $u = (\psi_y, -\psi_x)$ by the incompressibility condition in (1.1), where subscripts x and y denote the partial derivative for x and y, respectively. Using this function and newly denoting u as $\psi - \varphi$ we can rewrite the Eqs. (1.1) as

$$\begin{cases} \Delta^2 u + \Delta^2\varphi + R \cdot J(u+\varphi, \Delta(u+\varphi)) = (f_2)_x - (f_1)_y & \text{in } \Omega, \\ u = \dfrac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.1}$$

where J is a bilinear form defined by $J(u,v) = u_x v_y - u_y v_x$ and $\partial/\partial n$ stands for the normal derivative. Our aim is to verify the existence of a weak solution $u \in H_0^2(\Omega)$ of (2.1), where $H_0^2(\Omega) \equiv \{v \in H^2(\Omega) | v = \partial v/\partial n = 0 \text{ on } \partial\Omega\}$ with inner product $\langle u, v\rangle_{H_0^2} \equiv (\Delta u, \Delta v)_{L^2}$ for $u, v \in H_0^2(\Omega)$, and norm $\|u\|_{H_0^2(\Omega)} \equiv \|\Delta u\|_{L^2(\Omega)}$ for $u \in H_0^2(\Omega)$.

Let S_h be a finite dimensional subspace of $H_0^2(\Omega)$ that depends on h $(0<h<1)$. Usually S_h is taken to be a finite element subspace with mesh size h. We calculate an approximate solution $u_h \in C^1(\Omega)$ of (2.1) in the finite dimensional space, satisfying for all $v_h \in S_h$

$$(\Delta u_h + \Delta\varphi, \Delta v_h)_{L^2} + (R \cdot J(u_h+\varphi, \Delta(u_h+\varphi)), v_h)_{L^2} = ((f_2)_x - (f_1)_y, v_h)_{L^2}$$

and calculate $u_s \in C^2(\Omega)$ by smoothing of u_h. Then the linearized operator at u_s is represented as

$$\mathscr{L}u \equiv \Delta^2 u + R \cdot \{J(u_s+\varphi, \Delta u) + J(u, \Delta(u_s+\varphi))\},$$

and $\mathscr{L}$ is considered as the operator from $H_0^2(\Omega)$ to $H^{-2}(\Omega)$ in weak sense. We will verify the existence of the inverse $\mathscr{L}^{-1} : H^{-2}(\Omega) \to H_0^2(\Omega)$ and formulate the infinite dimensional Newton's method.

3. Invertibility of the linearized operator

By direct computations, we find that for any $q \in H^{-2}(\Omega)$ there exists a unique solution $v \in H_0^2(\Omega)$ satisfying

$$\begin{cases} \Delta^2 v = q & \text{in } \Omega, \\ v = \dfrac{\partial v}{\partial n} = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.1}$$

For $q \in H^{-2}(\Omega)$, let Kq be the unique solution $v \in H_0^2(\Omega)$ of the Eq. (3.1) then K is a compact operator from $H^{-1}(\Omega)$ to $H_0^2(\Omega)$. Using the compact operator on $H_0^2(\Omega)$

$$F_1(u) \equiv -R \cdot K\{J(u_s+\varphi, \Delta u) + J(u, \Delta(u_s+\varphi))\},$$

the equation $\mathscr{L}u = 0$ is equivalent to the fixed point equation $u = F_1(u)$. In order to show the invertibility of the linearized operator $\mathscr{L}$, by the Fredholm alternative, we only have to show the uniqueness of the solution of the equation $\mathscr{L}u = 0$.

Let $P_h : H_0^2(\Omega) \longrightarrow S_h$ denote the H_0^2-projection defined by

$$(\Delta(u - P_h u), \Delta v_h)_{L^2} = 0 \quad \text{for all } v_h \in S_h,$$

and we derive some error estimations for P_h. In what follows, we restrict ourselves to that the domain Ω is a unit square $(0,1)\times(0,1)$, and that S_h is the set of piecewise bicubic Hermite functions with uniform mesh on Ω (e.g., [5]). Note that then u_s is calculated as a cubic C^2-spline function with uniform mesh on Ω. However, our verification principle can also be applied to more general domains and approximation subspaces, when the appropriate a priori error estimates are obtained.

Table 1
Numerical value of constant C depending on h

$1/h$	20	40	60	80	100
C	0.7377	0.7811	0.8091	0.8278	0.8418

Concerning the error estimates for P_h we make use of the following lemma:

Lemma 1. *For $u \in H^4(\Omega) \cap H_0^2(\Omega)$ we have $\|u - P_h u\|_{H_0^2(\Omega)} \leqslant (Ch)^2 \|\Delta^2 u\|_{L^2(\Omega)}$, where C is a constant given in Table 1.*

Remark. The constant C in Lemma 1 was derived from the constructive error estimations with numerical computations for biharmonic problems, and it depends on each mesh size h as seen in Table 1. The basic idea for determination of the constant C is similar to the methods in [4,8]. We omit the proof of Lemma 1 here and will discuss it in the forthcoming paper [1] for details.

Now, as in [3,2], we decompose $u = F_1(u)$ into the finite and infinite dimensional parts

$$\begin{cases} P_h u = P_h F_1(u), \\ (I - P_h)u = (I - P_h)F_1(u). \end{cases} \tag{3.2}$$

Since we apply a Newton-like method only for the former part of (3.2), we define the following operator:

$$\mathcal{N}_h^1(u) \equiv P_h u - [I - F_1]_h^{-1}(P_h u - P_h F_1(u)),$$

where I is the identity map on $H_0^2(\Omega)$. And we assume that the restriction to S_h of the operator $P_h[I - F_1] : S_h \to S_h$ has the inverse $[I - F_1]_h^{-1}$. The validity of this assumption can be numerically confirmed in actual computations.

We next define the operator $T_1 : H_0^2(\Omega) \longrightarrow H_0^2(\Omega)$ by

$$T_1(u) \equiv \mathcal{N}_h^1(u) + (I - P_h)F_1(u).$$

Then T_1 becomes a compact map on $H_0^2(\Omega)$ and we have the following equivalence relation

$$u = T_1(u) \Longleftrightarrow u = F_1(u).$$

Our purpose is to find a unique fixed point of T_1 in a certain set $U \subset H_0^2(\Omega)$, which is called a 'candidate set'. Given positive real numbers γ and α we define the corresponding candidate set U by $U \equiv U_h \oplus [\alpha]$, where $U_h \equiv \{\phi_h \in S_h | \|\phi_h\|_{H_0^2} \leqslant \gamma\}$, $[\alpha] \equiv \{\phi_\perp \in S_\perp | \|\phi_\perp\|_{H_0^2} \leqslant \alpha\}$ and $S_\perp$ means the orthogonal complement of S_h in $H_0^2(\Omega)$. If the relation $\overline{T_1(U)} \subset \mathrm{int}(U)$ holds, by Schauder's fixed point theorem and the linearity of T_1, there exists a fixed point u of T_1 in U and the fixed point is unique, i.e., $u = 0$, which implies that the operator $\mathscr{L}$ is invertible. Decomposing $\overline{T_1(U)} \subset \mathrm{int}(U)$ into finite and infinite dimensional parts we have a sufficient condition for it as follows:

$$\begin{cases} \sup_{u \in U} \|\mathcal{N}_h^1(u)\|_{H_0^2(\Omega)} < \gamma, \\ \sup_{u \in U} \|(I - P_h)F_1(u)\|_{H_0^2(\Omega)} < \alpha. \end{cases} \tag{3.3}$$

We now derive the following theorem in which the verification condition (3.3) is numerically and simply described.

Theorem 1. *Let $\{\phi_i\}$ be the basis of S_h and define the following constants*:

$$C_0 = Ch, \quad C_1^s = \|\nabla(u_s + \varphi)\|_\infty, \quad C_2^s = \left\|\nabla \frac{\partial(u_s+\varphi)}{\partial x}\right\|_\infty + \left\|\nabla \frac{\partial(u_s+\varphi)}{\partial y}\right\|_\infty,$$

$$C_3^s = \|\nabla \varDelta(u_s + \varphi)\|_\infty, \quad C_p = \frac{1}{\pi\sqrt{2}}, \quad M_1 = \|L^T G^{-1} L\|_E,$$

$$K_1 = C_1^s + C_0^2 C_3^s, \quad K_2 = C_1^s + C_0 C_3^s C_p, \quad K_3 = \sqrt{2} C_1^s + C_p(C_2^s + C_0 C_3^s),$$

where C is the same constant as in Lemma 1, $\|\nabla v\|_\infty \equiv (\|\nabla v_x\|_\infty^2 + \|\nabla v_y\|_\infty^2)^{1/2}$, $\|\cdot\|_E$ denotes the matrix norm corresponding to the Euclidian vector norm, C_p is the Poincaré constant, the matrix $G = (G_{ij})$ is defined by $G_{ji} \equiv R(J(u_s + \varphi, \Delta\phi_i) + J(\phi_i, \varDelta(u_s + \varphi)), \phi_j)_{L^2(\Omega)} + (\Delta\phi_i, \Delta\phi_j)_{L^2(\Omega)}$, and $D = LL^T$ is a Cholesky decomposition of the matrix $D = (D_{ij})$ defined by $D_{ij} \equiv (\Delta\phi_i, \Delta\phi_j)_{L^2(\Omega)}$. For these constants, if the inequality

$$RC_0(K_1 + K_2 K_3 M_1 R C_0) < 1 \tag{3.4}$$

holds then the operator $\mathscr{L}$ is invertible.

Proof. We show sufficient conditions for (3.3). Denoting $u = u_1 + u_2,\ u_1 \in U_h,\ u_2 \in [\alpha]$, by some simple calculations we have $\mathscr{N}_h^1(u) = [I - F_1]_h^{-1} P_h F_1(u_2)$, and thus $\|\mathscr{N}_h^1(u)\|_{H_0^2(\Omega)} \leqslant M_1 \|P_h F_1(u_2)\|_{H_0^2(\Omega)}$ holds. (See [3,2] for details to such estimation.) Using error estimation in Lemma 1, we have $\|P_h F_1(u_2)\|_{H_0^2(\Omega)} \leqslant RC_0 K_3 \alpha$. Thus we derive a sufficient condition for the first inequality in (3.3) as

$$M_1 R C_0 K_3 \alpha < \gamma. \tag{3.5}$$

Now we estimate the left-hand side of the second inequality in (3.3). Noting that

$$\begin{aligned}\|(I - P_h)F_1(u)\|_{H_0^2(\Omega)} &\leqslant R\{\|(I - P_h)KJ(u_s + \varphi, \Delta u)\|_{H_0^2(\Omega)} + \|(I - P_h)KJ(u, \varDelta(u_s + \varphi))\|_{H_0^2(\Omega)}\}\\ &\leqslant RC_0 K_2 \gamma + RC_0 K_1 \alpha,\end{aligned}$$

we obtain the sufficient condition for the second inequality in (3.3) as

$$RC_0(K_1 \alpha + K_2 \gamma) < \alpha. \tag{3.6}$$

Combining the conditions (3.5) and (3.6) we finally obtain the sufficient condition for (3.3) as $RC_0(K_1 + K_2 K_3 M_1 R C_0) < 1$. $\square$

4. Verification procedure for nonlinear problem

In what follows we assume that the invertibility of the linearized operator $\mathscr{L}$ is confirmed by the method described in the previous section. We will verify the existence of solutions for (2.1) in the neighborhood of $u_X \in C^1(\Omega)$ satisfying $(\Delta u_X + \Delta\varphi, \Delta v_h)_{L^2(\Omega)} + (R \cdot J(u_s + \varphi, \varDelta(u_s + \varphi)), v_h)_{L^2(\Omega)} = ((f_2)_x - (f_1)_y, v_h)_{L^2(\Omega)}$ for all $v_h \in S_h$. Considering the function $\bar{u}$ satisfying

$$\begin{cases} \varDelta^2 \bar{u} = -\varDelta^2 \varphi - R \cdot J(u_s + \varphi, \varDelta(u_s + \varphi)) + (f_2)_x - (f_1)_y & \text{in } \Omega, \\ \bar{u} = \dfrac{\partial \bar{u}}{\partial n} = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.1}$$

and writing $w \equiv u - \bar{u},\ v_0 \equiv \bar{u} - u_X$, $u - u_X$ can be represented as $w + v_0$.

Noting that $u_X = P_h \bar{u}$, we see that $v_0 \in S_\perp$ and, by Lemma 1 the error estimate for v_0 can be derived

$$\|v_0\|_{H_0^2(\Omega)} \leqslant (Ch)^2 \| -\varDelta^2\varphi - R \cdot J(u_s + \varphi, \varDelta(u_s + \varphi)) + (f_2)_x - (f_1)_y\|_{L^2(\Omega)}.$$

Now we can rewrite (2.1) as

$$\begin{cases} \Delta^2 w = -R \cdot J(w + u_X + v_0 + \varphi, \Delta(w + u_X + v_0 + \varphi)) + R \cdot J(u_s + \varphi, \Delta(u_s + \varphi)) & \text{in } \Omega, \\ w = \dfrac{\partial w}{\partial n} = 0 \quad \text{on } \partial\Omega. \end{cases} \tag{4.2}$$

Thus defining the compact map on $H_0^2(\Omega)$: $F_2(w) \equiv RK\{J(u_s + \varphi, \Delta(u_s + \varphi)) - J(w + u_X + v_0 + \varphi, \Delta(w + u_X + v_0 + \varphi))\}$, we have the fixed point equation $w = F_2(w)$ which is equivalent to (4.2). Now we formulate the infinite dimensional Newton's method for this fixed point equation. Note that $w - [I - F_2'(-v_0 - u_X + u_s)]^{-1}(I - F_2)(w)$ can be equivalently represented as $\mathscr{L}^{-1}q(w)$, where $F_2'(-v_0 - u_X + u_s)$ stands for Fréchet derivative of F_2 at $-v_0 - u_X + u_s$ and $q(w) \equiv R\{J(u_s + \varphi, \Delta(u_s + \varphi)) - J(w + u_X + v_0 + \varphi, \Delta(w + u_X + v_0 + \varphi)) + J(u_s + \varphi, \Delta w) + J(w, \Delta(u_s + \varphi))\}$. Then it is seen that $w = F_2(w) \Longleftrightarrow w = T_2(w)$, where $T_2(w) \equiv \mathscr{L}^{-1}q(w)$ is a compact map on $H_0^2(\Omega)$.

We intend to find a fixed point of T_2 in a set W defined by $W \equiv \{w \in H_0^2(\Omega) | \|w\|_{H_0^2(\Omega)} \leqslant \alpha\}$, where α is a positive number. If the relation $T_2(W) \subset W$ holds, by Schauder's fixed point theorem there exists a fixed point of T_2 in W. In order to derive a sufficient condition for $T_2(W) \subset W$, we first prepare for the following constants:

$$\kappa \equiv C_0 R(K_1 + K_2 K_3 M_1 C_0 R), \quad \tau_1 = \frac{C_0 R M_1 K_2}{1 - \kappa}, \quad \tau_2 = \frac{1}{1 - \kappa},$$

$$\tau_3 = M_1(C_0 R K_3 \tau_1 + 1), \quad \tau_4 = M_1 C_0 R K_3 \tau_2, \quad b = \|v_0\|_{H_0^2(\Omega)}, \quad C_4 = \frac{1}{\pi},$$

where C_4 is an embedding constant satisfying $\|\nabla u\|_{L^4(\Omega)} \leqslant C_4 \|\Delta u\|_{L^2(\Omega)}$ for $u \in H_0^2(\Omega)$ and we have used the optimal embedding estimates $C_4 = 1/\pi$ which can be derived by the result in [6]. Moreover for a matrix

$$S = \begin{pmatrix} \tau_1^2 + \tau_3^2 & \tau_1\tau_2 + \tau_3\tau_4 \\ \tau_1\tau_2 + \tau_3\tau_4 & \tau_2^2 + \tau_4^2 \end{pmatrix}$$

we set $M_2 \equiv \|S\|_E^{1/2}$ and define the following constants:

$$C_1^X = \|\nabla(u_X + \varphi)\|_\infty, \quad C_2^X = \left\| \nabla \frac{\partial(u_X + \varphi)}{\partial x} \right\|_\infty + \left\| \nabla \frac{\partial(u_X + \varphi)}{\partial y} \right\|_\infty,$$

$$C_3^X = \|\Delta(u_X + \varphi)\|_\infty, \quad D_1^\delta = \|\nabla(u_X - u_s)\|_{L^2(\Omega)},$$

$$D_2^\delta = \|J(u_X - u_s, \Delta(u_s + \varphi))\|_{L^2(\Omega)}, \quad D_3^\delta = \|\Delta(u_X - u_s)\|_{L^2(\Omega)}.$$

Since a sufficient condition for $T_2(W) \subset W$ is $\sup_{w \in W} \|T_2(w)\|_{H_0^2(\Omega)} \leqslant \alpha$, by estimating the left hand side of this inequality, we can derive the following theorem.

Theorem 2. *Assume that the invertibility condition* (3.4) *holds. Using the same constants in Theorem* 1, *if there exists a real number* $\alpha > 0$ *satisfying the quadratic inequality in* α : $M_2 R\{C_4^2(\alpha + b)^2 + C_4^2 \alpha D_3^\delta + C_3^X C_p C_0 b + \alpha D_1^\delta C_p + C_0 b(\sqrt{2} C_1^X + C_p C_3^X) + C_p^2 D_2^\delta + C_p C_1^X D_3^\delta\} \leqslant \alpha$, *then there exists a fixed point of* T_2 *in* W.

Proof. For $q(w) \in H^{-2}(\Omega)$ consider the solution $\phi \in H_0^2(\Omega)$ of the problem

$$\begin{cases} \mathscr{L}\phi = q(w) & \text{in } \Omega, \\ \phi = \dfrac{\partial \phi}{\partial n} = 0 \quad \text{on } \partial\Omega. \end{cases} \tag{4.3}$$

Then writing $\phi = \phi_h + \phi_\perp, \ \phi_h \in S_h, \ \phi_\perp \in S_\perp$, we have

$$\begin{cases} \|\phi_h\|_{H_0^2(\Omega)} \leqslant M_1 R C_0 K_3 \|\phi_\perp\|_{H_0^2(\Omega)} + M_1 \|P_h K q(w)\|_{H_0^2(\Omega)}, \\ \|\phi_\perp\|_{H_0^2(\Omega)} \leqslant R C_0 (K_1 \|\phi_\perp\|_{H_0^2(\Omega)} + K_2 \|P_h K q(w)\|_{H_0^2(\Omega)}) + \|(I - P_h) K q(w)\|_{H_0^2(\Omega)}. \end{cases} \tag{4.4}$$

Table 2
Verification results for driven cavity problem ($h = 1/75$)

R	M_1	M_2	$\|v_0\|_{H_0^2(\Omega)}$	D_3^δ	α
100	1.1746	1.5841	6.9318e−4	3.8268e−6	5.9094e−4
200	1.1945	3.1510	6.9313e−4	4.8677e−6	3.2670e−3

Noting that $\kappa < 1$ holds because of the invertibility of $\mathscr{L}$, we have

$$\begin{cases} \|\phi_h\|_{H_0^2(\Omega)} \leqslant \tau_3 \|P_h K q(w)\|_{H_0^2(\Omega)} + \tau_4 \|(I - P_h) K q(w)\|_{H_0^2(\Omega)}, \\ \|\phi_\perp\|_{H_0^2(\Omega)} \leqslant \tau_1 \|P_h K q(w)\|_{H_0^2(\Omega)} + \tau_2 \|(I - P_h) K q(w)\|_{H_0^2(\Omega)}. \end{cases} \tag{4.5}$$

Therefore by some simple calculations, using (4.4) and (4.5) we obtain

$$\|\phi\|_{H_0^2(\Omega)} \leqslant M_2 \|K q(w)\|_{H_0^2(\Omega)} \leqslant M_2 \|q(w)\|_{H^{-2}}. \tag{4.6}$$

Furthermore, we have the estimations

$$\begin{aligned} \|q(w)\|_{H^{-2}} &= \sup_{\theta \in H_0^2(\Omega), \|\theta\|_{H_0^2(\Omega)} = 1} |\langle q(w), \theta \rangle_{H^{-2}, H_0^2}| \\ &\leqslant R\{C_4^2(\alpha + b)^2 + C_4^2 \alpha D_3^\delta + C_3^X C_p C_0 b + \alpha D_1^\delta C_p + C_0 b(\sqrt{2} C_1^X + C_p C_3^X) \\ &\quad + C_p^2 D_2^\delta + C_p C_1^X D_3^\delta\}, \end{aligned}$$

where $\langle \cdot, \cdot \rangle_{H^{-2}, H_0^2}$ means the canonical duality pairing. Thus we obtain

$$\begin{aligned} \|\mathscr{L}^{-1} q(w)\|_{H_0^2(\Omega)} \leqslant M_2 R\{\|C_4^2(\alpha + b)^2 + C_4^2 \alpha D_3^\delta + C_3^X C_p C_0 b + \alpha D_1^\delta C_p \\ + C_0 b(\sqrt{2} C_1^X + C_p C_3^X) + C_p^2 D_2^\delta + C_p C_1^X D_3^\delta\} \end{aligned}$$

and the desired assertion is proved. □

5. Numerical examples

Particularly, we consider the two dimensional driven cavity problem with $f = 0$ and $(\varphi_y, -\varphi_x) = g$ in (1.1), where $\varphi(x, y) = x^2(1 - x)^2 y^2 (1 - y)$.

The computations were carried out on the DELL Precision WorkStation 650 (Intel Xeon 3.2 GHz) using MATLAB (Ver. 6.5.1). The verification results are shown in Table 2, and the solution u in (2.1) is enclosed as

$$\|u - u_X\|_{H_0^2(\Omega)} \leqslant \|v_0\|_{H_0^2(\Omega)} + \alpha.$$

It seems that Wieners' method would not be able to apply to the Reynolds number higher than 20 in [7]. On the other hand, we enclosed the stationary solution for the Reynolds number up to 200, and our method can be applied, in principle, to higher Reynolds numbers by using more accurate approximation subspaces, i.e., smaller mesh sizes.

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